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SUMMARY

Analysis and experiment are presented for the wrinkling behavior of stretched membranes subject to a torque loading through an attached hub. The analysis makes use of a theory for partly wrinkled membranes which is based on a study of average deformations in the wrinkled region. Closed-form solutions are obtained for several different boundary conditions, and results are given in the form of torque-rotation plots. Experimental results from tests on thin sheets of plastic film were found to be in very good agreement with theory.

INTRODUCTION

In the design of space-vehicle structures there are applications for very thin walled shell structures - shells which are so thin that they can be treated analytically as membranes which have zero bending stiffness and can carry no compressive stress. Under certain loading conditions, wrinkling can occur over a portion of a membrane structure, and it is desirable to understand the behavior of the structure in such a condition. The attainment of this goal is advanced by the solution of fundamental wrinkled-membrane problems; one such problem is that of the rotation of a hub attached to a flat stretched membrane. This particular problem will be of importance in itself when structural members or other components must be attached to thin membrane-like walls.

The somewhat related problem of an annular plate buckled by the rotation of an attached hub has been solved by W. R. Dean in reference 1. In reference 2 Reissner has solved the case of an initially unstretched membrane by using tension-field theory. In reference 3 Stein and Hedgepeth present a theory for partly wrinkled membranes, and the solution is given for the rotation of a hub attached to a stretched membrane of infinite extent. In this wrinkled membrane theory a detailed study of the wrinkles is not made, but average strains and displacements in the wrinkled region are considered. This theory is

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*The information presented herein is based in part upon a thesis offered in partial fulfillment of the requirements for the degree of Master of Science in Engineering Mechanics, Virginia Polytechnic Institute, Blacksburg, Virginia, June 1964.

limited to small average strains and displacements in the same sense as in linear elasticity theory.

In the present paper, a generalization of the results given in reference 3 is presented, and an experimental study is described which lends credence to the validity of partly wrinkled membrane theory. The primary problem solved in the present paper is that of the rotation of a hub attached to a finite, stretched, circular membrane, in which it is assumed that the hub is attached subsequent to the application of a uniform tension in the membrane. The special case of the rotation of a hub on a circular membrane with zero tension is included in appendix A. For this special case the equations in the present paper reduce to the equations given by Reissner in reference 2. Finally, the case is treated in which the hub is attached prior to stretching the finite membrane; results obtained for an infinite membrane are discussed in appendix B.

SYMBOLS

a	radius of hub
b	radius of circular membrane
r	radial coordinate
r_o	radius of plate as defined in figure 11
t	thickness of membrane
u,v	displacements in r and θ directions
x,y	rectangular coordinates
E	Young's modulus
G	shear modulus, $\dfrac{E}{2(1 + \nu)}$
M	torque
P	load per unit thickness
R	radial extent of wrinkled region
T	initial uniform tensile stress
N	number of loads
$C_1,C_2,\ldots$	constants
α,β	angles defined in figure 9

θ angular coordinate

λ function determining strain in direction perpendicular to wrinkles, "variable Poisson's ratio"

ν Poisson's ratio for material

ϕ rotation of hub

ϵ_1, ϵ_2 principal strains

ϵ_x, ϵ_y direct strains in rectangular-coordinate system

γ_{xy} shear strain in rectangular-coordinate system

$(\rho_1, \phi_1), (\rho_2, \phi_2)$ polar coordinates (see fig. 11)

r, ϵ_θ direct strains in polar-coordinate system

$r\theta$ shear strain in polar-coordinate system

$1, \sigma_2$ principal stresses

x, σ_y direct stresses in rectangular-coordinate system

xy shear stress in rectangular-coordinate system

r, σ_θ direct stresses in polar-coordinate system

$r\theta$ shear stress in polar-coordinate system

WRINKLING THEORY

By definition a membrane has zero bending stiffness and, therefore, can carry no compressive load. In reference 3 this feature is utilized as a basis for developing a theory for membranes which are wrinkled over a portion of their surface. In this theory it is reasoned that for wrinkling to occur, one principal stress must be zero and the other nonzero. The zero principal stress is perpendicular to the wrinkles while the nonzero principal stress acts along the wrinkles. From the plane stress equations for principal stresses, the condition that one principal stress vanish is given by

$$\sigma_x \sigma_y = \tau_{xy}^2 \tag{1}$$

Equation (1) along with the equilibrium equations

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned}\right\} \tag{2}$$

form a set of three equations in the three unknown stresses for the wrinkled region.

The strain along the wrinkles is given as

$$\epsilon_1 = \frac{\sigma_1}{E} \tag{3}$$

while perpendicular to the wrinkles the average strain is considered to be

$$\epsilon_2 = -\lambda(x,y)\frac{\sigma_1}{E} \tag{4}$$

The quantity $\lambda(x,y)$ is introduced as the "variable Poisson's ratio" to allow for contraction of the material in a direction normal to the wrinkles. At the boundary between wrinkled and unwrinkled regions of the membrane, $\lambda(x,y)$ must equal Poisson's ratio for the material.

The stress-strain relations may now be written for the wrinkled region as

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}\left(\sigma_x - \lambda\sigma_y\right) \\ \epsilon_y &= \frac{1}{E}\left(\sigma_y - \lambda\sigma_x\right) \\ \gamma_{xy} &= \frac{2(1 + \lambda)}{E}\,\tau_{xy} \end{aligned}\right\} \tag{5}$$

The usual strain-displacement equations are

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{6}$$

where the displacements in the wrinkled region are considered to be average displacements. A detailed study of the wrinkles has not been attempted in this theory since the overall behavior of the membrane structure is of primary interest.

ANALYSIS

In the problem treated in the present paper, the membrane is stretched by a uniform stress T, and the attached hub is rotated through an angle ϕ by a torque M. After a certain value of torque, wrinkles begin to form around the hub out to some radius R as illustrated schematically in figure 1 and shown in the photograph in figure 2. To analyze this behavior, the wrinkling theory previously discussed must be used. The solution for the unwrinkled region can be obtained readily by a plane stress elasticity analysis. Since the wrinkling theory involves average deformations, stresses, strains, and displacements for this problem are considered independent of θ (radially symmetric).

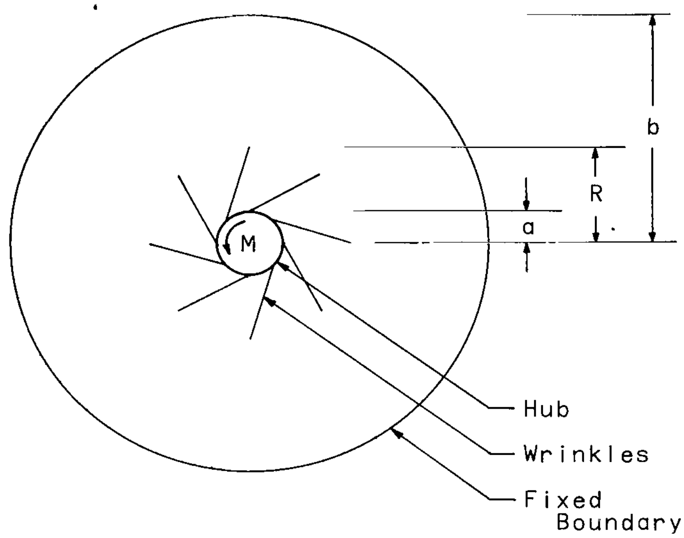

Figure 1.- Schematic diagram of a membrane wrinkled by rotation of attached hub.

Basic Equations

The equilibrium equations for a radially symmetric stress state are

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{7}$$

$$\frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} = 0 \tag{8}$$

Integration of equation (8) gives

$$\tau_{r\theta} = -\frac{M}{2\pi r^2 t} \tag{9}$$

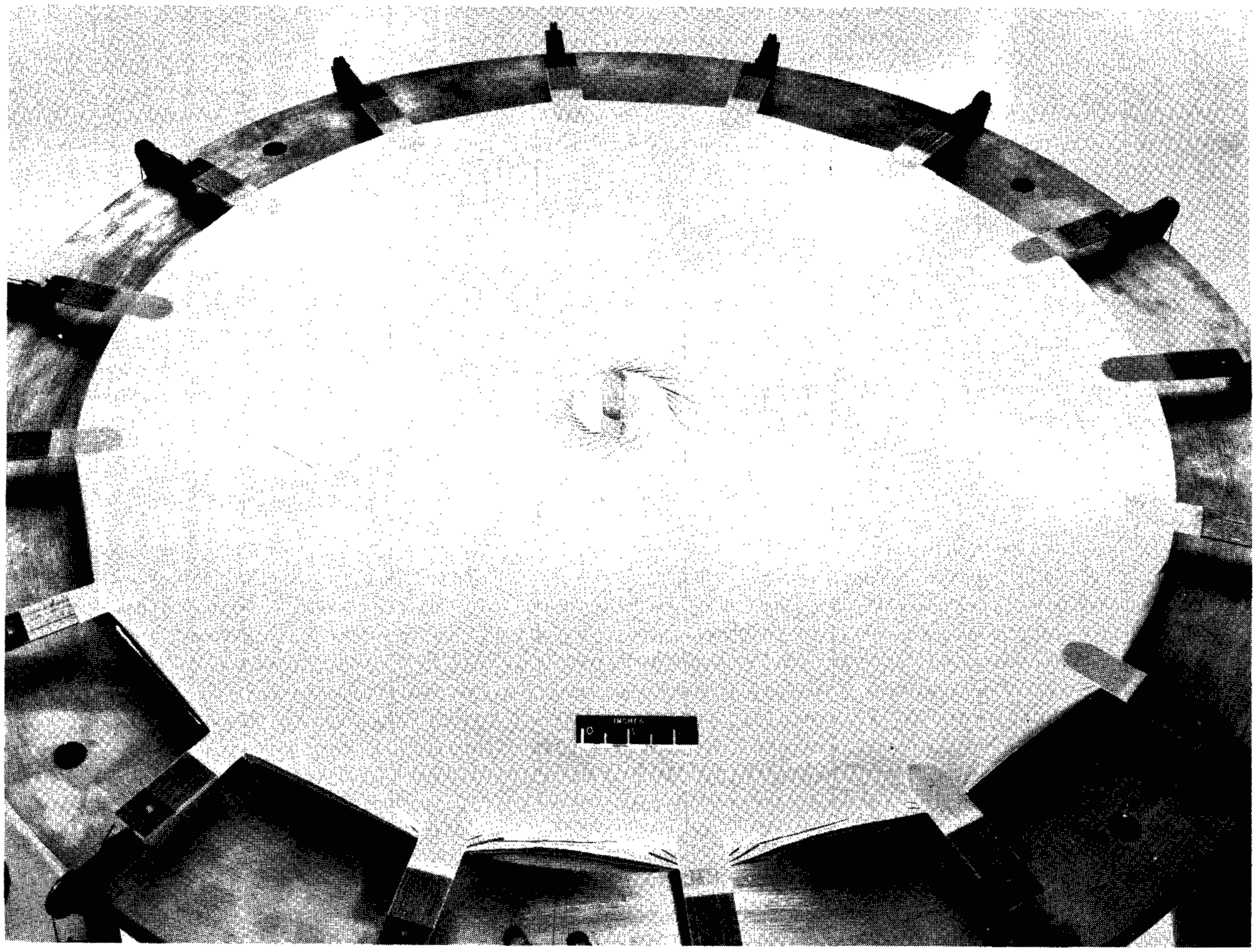

Figure 2.- Stretched specimen in wrinkled condition. L-63-1974

where the constant of integration has been chosen to satisfy the shear-stress-torque relationship

$$M = -t \int_0^{2\pi} \tau_{r\theta} r^2 d\theta$$

The strain-displacement relations for radially symmetric deformations are

$$\left.\begin{aligned} \epsilon_r &= \frac{du}{dr} \\ \epsilon_\theta &= \frac{u}{r} \\ \gamma_{r\theta} &= \frac{dv}{dr} - \frac{v}{r} \end{aligned}\right\} \tag{10}$$

Thus, compatibility of the strains requires that

$$\epsilon_r = \frac{d}{dr}(r\epsilon_\theta) \tag{11}$$

Solution of Equations

Unwrinkled region.- In the unwrinkled outer region of the circular membrane $(r > R)$, the following conventional stress-strain relations hold:

$$\left.\begin{aligned} \epsilon_r &= \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) \\ \epsilon_\theta &= \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) \\ \gamma_{r\theta} &= \frac{\tau_{r\theta}}{G} \end{aligned}\right\} \tag{12}$$

Elimination of σ_θ between the first equilibrium equation (7) and equations (11) and (12) gives the following differential equation for σ_r

$$r\,\frac{d^2\sigma_r}{dr^2} + 3\,\frac{d\sigma_r}{dr} = 0$$

The solution of this differential equation is

$$\sigma_r = \frac{C_1}{r^2} + C_2 \tag{13}$$

From equation (7) σ_θ is obtained as:

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 \tag{14}$$

To determine the displacements, equations (13) and (14) are substituted into the stress-strain relations (eqs. (12)) and then the strain-displacement relations (eqs. (10)) are utilized to obtain

$$\left.\begin{aligned} \frac{u}{r} &= \frac{1}{E}\left[-C_1\,\frac{(1 + \nu)}{r^2} + (1 - \nu)C_2\right] \\ &\text{and} \\ \frac{dv}{dr} - \frac{v}{r} &= -\frac{1}{G}\,\frac{M}{2\pi r^2 t} \end{aligned}\right\} \tag{15}$$

The solution of the second of equations (15) is

$$v = \frac{M}{4G\pi rt} + C_3 r \tag{16}$$

Wrinkled region.- In the inner region, $a < r < R$, the counterpart of the condition for zero principal stress (eq. (1)) in polar coordinates is

$$\sigma_r \sigma_\theta = \tau_{r\theta}^{\ 2} \tag{17}$$

Equation (17) together with equation (9) gives

$$\sigma_\theta = \frac{M^2}{4\pi^2 t^2 r^4} \frac{1}{\sigma_r} \tag{18}$$

Equation (18) together with the first equilibrium equation (eq. (7)) may now be written as

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r}{r} - \frac{M^2}{4\pi^2 t^2 r^5} \frac{1}{\sigma_r} = 0 \tag{19}$$

Equation (19) may be rewritten as

$$r\sigma_r \frac{d(r\sigma_r)}{dr} = \frac{M^2}{4\pi^2 t^2} \frac{1}{r^3}$$

and after multiplying this equation through by dr, the quantity $r\sigma_r$ is obtained by integration. The solution is

$$\sigma_r = \frac{1}{r}\sqrt{C_4 - \frac{M^2}{4\pi^2 t^2 r^2}} \tag{20}$$

and from equation (18)

$$\sigma_\theta = \frac{M^2}{4\pi^2 t^2 r^3} \frac{1}{\sqrt{C_4 - \frac{M^2}{4\pi^2 t^2 r^2}}} \tag{21}$$

The stress-strain relations for the wrinkled region are

$$\left.\begin{aligned} \epsilon_r &= \frac{1}{E}\left(\sigma_r - \lambda\sigma_\theta\right) \\ \epsilon_\theta &= \frac{1}{E}\left(\sigma_\theta - \lambda\sigma_r\right) \\ \gamma_{r\theta} &= \frac{2(1 + \lambda)}{E}\tau_{r\theta} \end{aligned}\right\} \tag{22}$$

where $\lambda = \lambda(r)$ is the variable Poisson's ratio as previously discussed. From the compatibility equation (11), the first equilibrium equation (7), and equations (22), the following differential equation is obtained:

$$\frac{d\lambda}{dr} = \frac{1}{r\sigma_r}\frac{d}{dr}\left(r\sigma_\theta\right) - \frac{1}{r}$$

After substitution for σ_r and σ_θ from equations (20) and (21) in the preceding equation, integration yields the following equation for λ:

$$\lambda = \frac{1}{2}\frac{1}{\dfrac{4\pi^2t^2C_4r^2}{M^2} - 1} - \frac{1}{2}\log\left(\frac{4\pi^2t^2C_4r^2}{M^2} - 1\right) + C_5 \tag{23}$$

Now, from the strain-displacement equations (10) the displacements for the wrinkled region may be written as

$$u = \frac{M}{4\pi Etr}\sqrt{\frac{4\pi^2t^2C_4r^2}{M^2} - 1}\left[\frac{1}{\dfrac{4\pi^2t^2C_4r^2}{M^2} - 1} + \log\left(\frac{4\pi^2t^2C_4r^2}{M^2} - 1\right) - 2C_5\right] \tag{24}$$

and

$$v = \frac{M}{4\pi Etr}\left[1 + 2C_5 - \log\left(\frac{4\pi^2t^2C_4r^2}{M^2} - 1\right) + C_6r^2\right] \tag{25}$$

To determine the seven unknown constants $\left(C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6,\right.$ and $\left.R\right)$ seven relations are necessary. Four relations are obtained from enforcing continuity on σ_r and the displacements u and v at the border between the wrinkled and unwrinkled regions and from the continuity condition that $\lambda = \nu$ at $r = R$. The three remaining relations are obtained from boundary conditions on u and v at the outer boundary and a boundary condition on

u at the hub. Note that one additional constant of integration has already been determined in relating the torque to the shear stress (eq. (9)).

Continuity conditions between wrinkled and unwrinkled regions.- By using the condition $\lambda = \nu$ at $r = R$ in equation (23)

$$C_5 = \nu - \frac{1}{2} \frac{1}{\frac{4\pi^2 t^2 C_4 R^2}{M^2} - 1} + \frac{1}{2} \log\left(\frac{4\pi^2 t^2 C_4 R^2}{M^2} - 1\right) \tag{26}$$

For σ_r to be continuous at $r = R$

$$\frac{M^2}{4\pi^2 t^2 R^2} + \left(\frac{C_1}{R} + C_2 R\right)^2 - C_4 = 0 \tag{27}$$

Two more relations between constants are obtained from the conditions that the u and v displacements be continuous at $r = R$; thus

$$\left(\frac{2\pi t C_1}{M}\right)^2 = \frac{4\pi^2 t^2 C_2{}^2 R^4}{M^2} - 1 \tag{28}$$

and

$$C_6 = \frac{1}{R^2}\left(1 + \frac{4\pi E t C_3 R^2}{M} + \frac{1}{\frac{4\pi^2 t^2 C_4 R^2}{M^2} - 1}\right) \tag{29}$$

Conditions at edge of hub and outer boundary of membrane.- At the outer boundary the tangential displacement is taken as zero $(v(b) = 0)$. For a membrane stretched by a uniform tension T, the radial displacement is $u(r) = (1 - \nu)\frac{Tr}{E}$, thus when the hub is attached to a finite circular membrane after the membrane is stretched, the boundary conditions on the radial displacement u are:

$$u(b) = \frac{(1 - \nu)Tb}{E}$$

$$u(a) = \frac{(1 - \nu)Ta}{E}$$

When the boundary condition $v(b) = 0$ is applied to equation (16), the constant C_3 is determined immediately as:

$$C_3 = -\frac{M}{4G\pi b^2 t} \tag{30}$$

When the two boundary conditions on u are applied to the first of equations (15) and to equation (24), respectively, the following two relations are obtained:

$$C_2 = T + \frac{C_1}{b^2}\frac{(1 + \nu)}{(1 - \nu)} \tag{31}$$

and

$$\frac{(1 - \nu)Ta}{E} = \frac{M}{4\pi Eta}\sqrt{\frac{4\pi^2 t^2 C_4 a^2}{M^2} - 1}\left[\frac{1}{\frac{4\pi^2 t^2 C_4 a^2}{M^2} - 1} + \log\left(\frac{4\pi^2 t^2 C_4 a^2}{M^2} - 1\right) - 2C_5\right] \tag{32}$$

If the torque M on the hub is small enough so that wrinkling does not occur, equation (16) holds throughout the membrane so that

$$v = \frac{M}{4G\pi t}\left(\frac{1}{r} - \frac{r}{b^2}\right)$$

where C_3 has been evaluated from equation (30). Thus, for the prewrinkled range, the hub rotation $\phi = \frac{v(a)}{a}$ can be expressed as a linear function of M as follows:

$$\frac{2G\phi}{T\left(1 - \frac{a^2}{b^2}\right)} = \frac{M}{2\pi a^2 Tt} \tag{33}$$

When the torque on the hub is increased beyond the value at which wrinkling occurs, the solution proceeds as discussed subsequently. Let $\nu = 1/3$ and define the following dimensionless parameters:

$$\overline{M} = \frac{M}{2\pi a^2 Tt}, \quad \overline{\phi} = \frac{2G\phi}{T\left(1 - \frac{a^2}{b^2}\right)}, \quad \overline{C_1} = \frac{C_1}{a^2 T}, \quad \overline{C_4} = \frac{C_4}{a^2 T^2}, \quad \overline{R} = \frac{R}{a}$$

Combine equations (26) and (32) to obtain

$$\bar{M}\sqrt{\frac{\bar{C}_4}{\bar{M}^2} - 1}\left[\frac{1}{\frac{\bar{C}_4}{\bar{M}^2} - 1} + \frac{1}{\bar{R}^2 \frac{\bar{C}_4}{\bar{M}^2} - 1} - \log\left(\frac{\bar{R}^2 \frac{\bar{C}_4}{\bar{M}^2} - 1}{\frac{\bar{C}_4}{\bar{M}^2} - 1}\right) - \frac{2}{3}\right] = \frac{4}{3} \tag{34}$$

Combine equations (31) and (28) to obtain

$$\bar{C}_1{}^2 - \left[1 + 2\bar{C}_1\left(\frac{a}{b}\right)^2\right]^2 \bar{R}^4 + \bar{M}^2 = 0 \tag{35}$$

Combine equations (31) and (27) to obtain

$$\bar{C}_4 = \left[\bar{R} + \left(\frac{1}{\bar{R}} + 2\bar{R}\,\frac{a^2}{b^2}\right)\bar{C}_1\right]^2 + \frac{\bar{M}^2}{\bar{R}^2} \tag{36}$$

The quantities $\bar{M}$, $\bar{C}_1$, and $\bar{C}_4$ are found from equations (34), (35), and (36) for given values of $\bar{R}$ and a/b by trial and error.

The rotation of the hub $\phi = \frac{v(a)}{a}$ is found from equation (25) where C_6 is eliminated by using equation (29), and C_5 is eliminated from equation (26). Equation (25) can be written as

$$\bar{\phi} = \frac{\bar{M}}{\frac{8}{3}\left(1 - \frac{a^2}{b^2}\right)}\left[\frac{\frac{1}{\bar{R}^2 - 1}}{\bar{R}^2 \frac{\bar{C}_4}{\bar{M}^2} - 1} + \log\left(\frac{\bar{R}^2 \frac{\bar{C}_4}{\bar{M}^2} - 1}{\frac{\bar{C}_4}{\bar{M}^2} - 1}\right) + \frac{1}{\bar{R}^2} - \frac{8}{3}\,\frac{a^2}{b^2} + \frac{5}{3}\right] \tag{37}$$

A nondimensional plot of this torque-rotation relationship is presented in figure 3 for values of a/b equal to 1/2, 5/32, 3/32, and 0. Note that for a/b = 0 the solution in this paper reduces to the solution given in reference 3 for an infinite membrane.

The maximum stress in the membrane is the nonzero principal stress at the hub. In the wrinkled region this principal stress may be written as

$$\sigma_1 = \sigma_r + \sigma_\theta \tag{38}$$

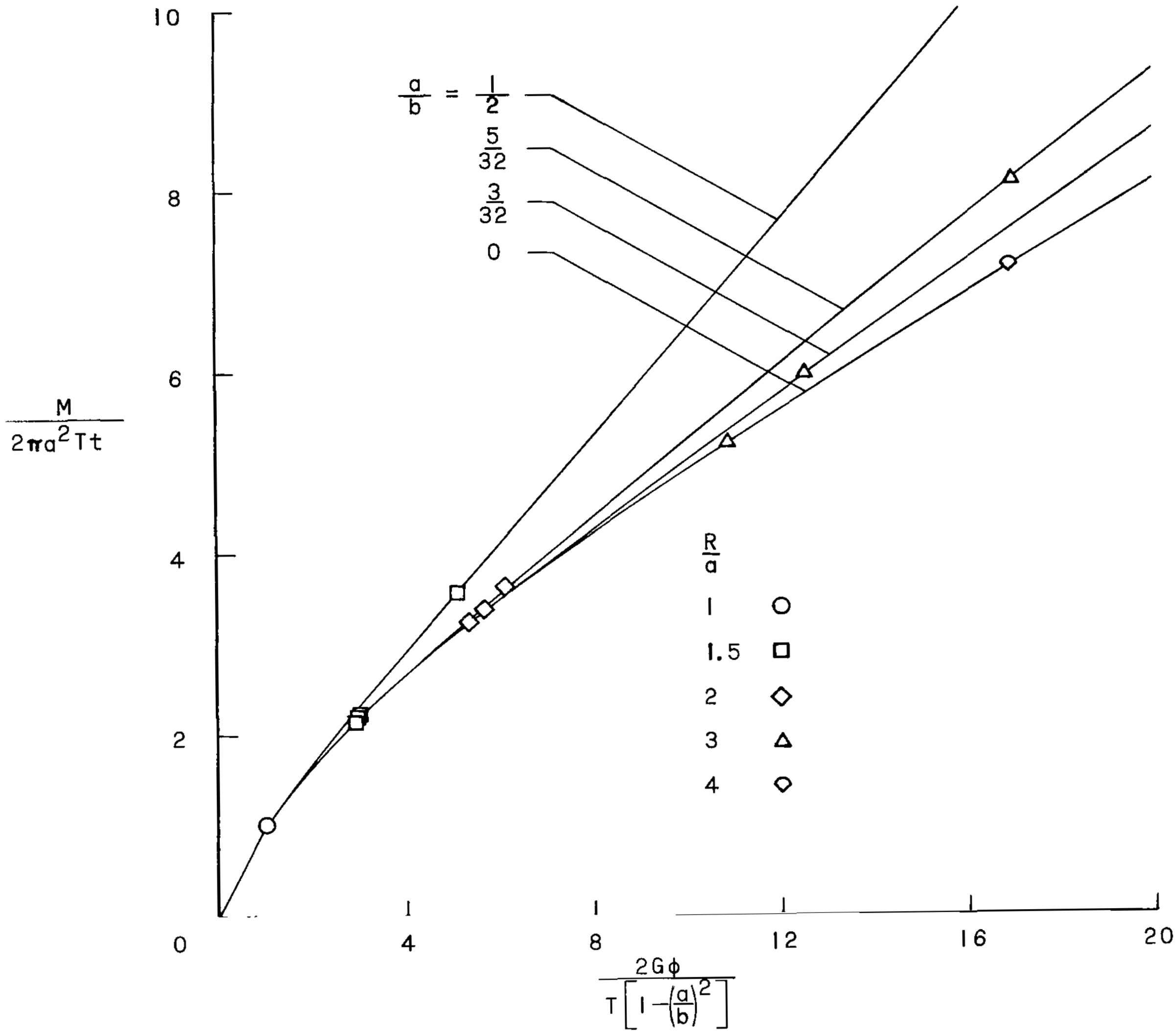

Figure 3.- Plot of rotation due to torque of hub on flat stretched membrane.

When the expressions for stresses from equations (20) and (21) are substituted into equation (38), the following equation for σ_1 is obtained:

$$\sigma_1 = \frac{C_4/r}{\sqrt{C_4 - \frac{M^2}{4\pi^2 t^2 r^2}}} \tag{39}$$

Now, by evaluating σ_1 at the hub $(r = a)$ and using the dimensionless parameters previously described, the following nondimensional equation can be written:

$$\frac{\sigma_1(a)}{T} = \frac{\bar{C}_4}{\sqrt{\bar{C}_4 - \bar{M}^2}} \tag{40}$$

A plot of $\sigma_1(a)/T$ as a function of $\bar{M}$ is presented in figure 4 for values of a/b equal to 1/2, 5/32, 3/32, and 0.

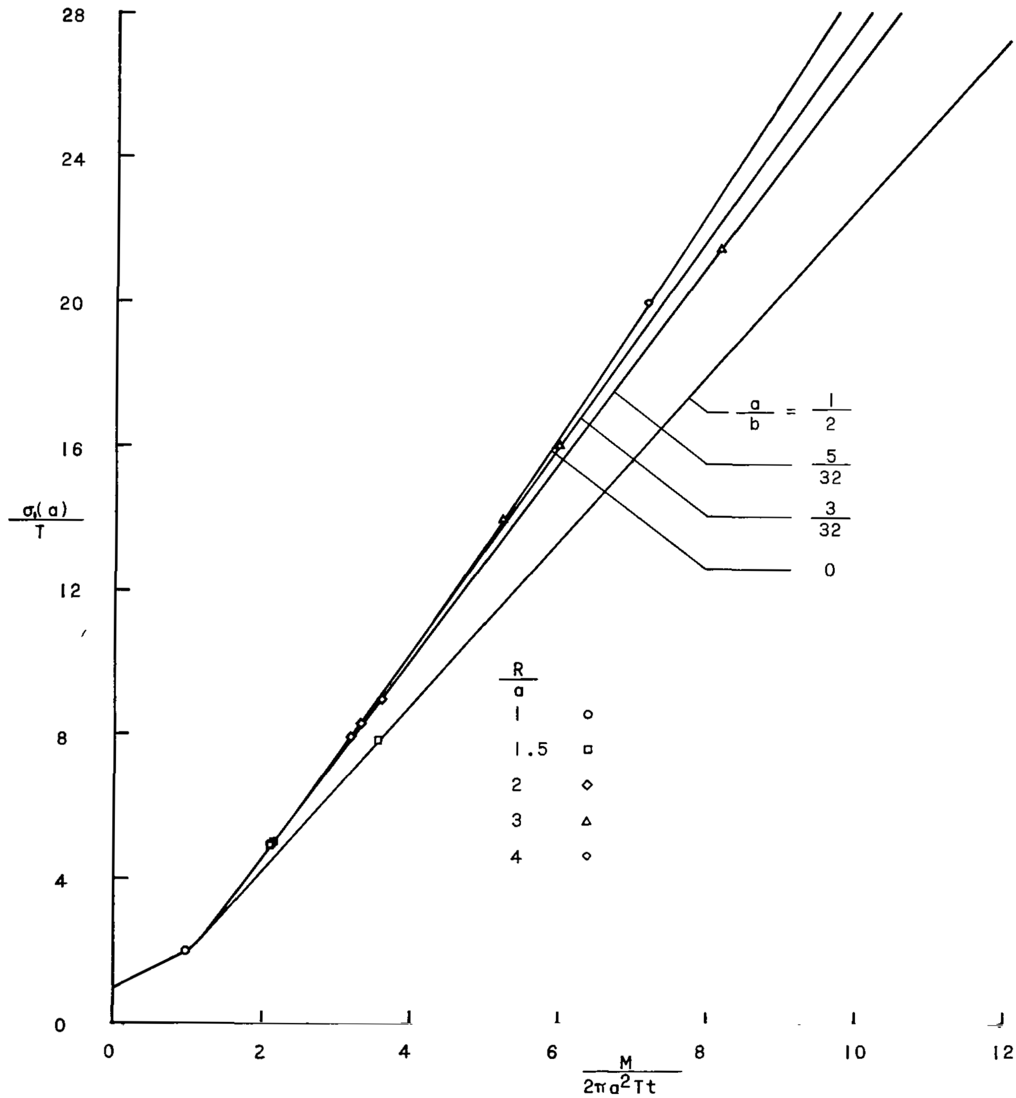

Figure 4.- Plot of principal stress at hub.

EXPERIMENT

The units used for the physical quantities in this section are given both in the U.S. Customary Units and in the International System of Units, SI (ref. 4). For the purpose of explaining the relationships between these two systems of units, appendix C is included.

Test Specimens

Tests were performed on circular specimens of 1/2-mil (12.7-μm) polyethylene terephthalate plastic film, one of which is shown in figure 2 mounted on the test fixture. To achieve an initial uniform stress T in a portion of the specimen, 16 evenly spaced radial loads were applied at the periphery of the specimen. An analysis (see appendix D) was made which indicated that under such a loading a uniform stress state would exist in the central portion of the specimen, extending outward about six-tenths of the distance from the center. Based on this result the specimens were chosen to be 28 inches (71.1 cm) in diameter with a boundary ring 16 inches (40.6 cm) in diameter.

To transmit the radial load as smoothly as possible into the thin plastic film specimen, filament-reinforced pressure-sensitive adhesive-tape tabs were attached as can be seen in the photograph in figure 2. The tape tabs were rounded to reduce the effect of stress concentrations.

Test Apparatus and Procedure

A schematic drawing of the test setup and rotation-measuring apparatus is shown in figure 5. A photograph of the test fixture before mounting a specimen is shown in figure 6. The procedure for mounting a specimen was as follows:

(1) A thin coat of a synthetic elastomeric adhesive was applied to the hub, to the boundary ring, and to the portion of the specimen which would come in contact with the hub and boundary.

(2) A plexiglass ring was used to keep the specimen raised off the hub and boundary ring until the specimen was stretched. This ring was mounted on screws and was placed concentric with the boundary ring on the text fixture (see fig. 6). The plexiglass ring was then adjusted until it was parallel with the top of the boundary ring but slightly higher.

(3) The specimen was then placed over the plexiglass ring and the 16 radial loads were applied through strings which passed over knife-edge supported pulleys and were attached to separate shot buckets of equal weight (see fig. 7).

(4) It was found that after the specimen was completely loaded, due to the very low constraints offered by the knife-edge pulleys, the system was self-centering. Therefore, to complete the mounting of the specimen, the plexiglass ring was simply lowered until the glued portion of the specimen came in contact with the boundary ring and hub, at which time a completed seal was accomplished by applying pressure to the glued joints with a cotton swab. The 16 radial loads were not removed from the specimen after gluing to help insure a fixed boundary.

The torque was applied through a pulley which was attached to a shaft which, in turn, was attached to the hub. The shaft was supported by ball bearings to

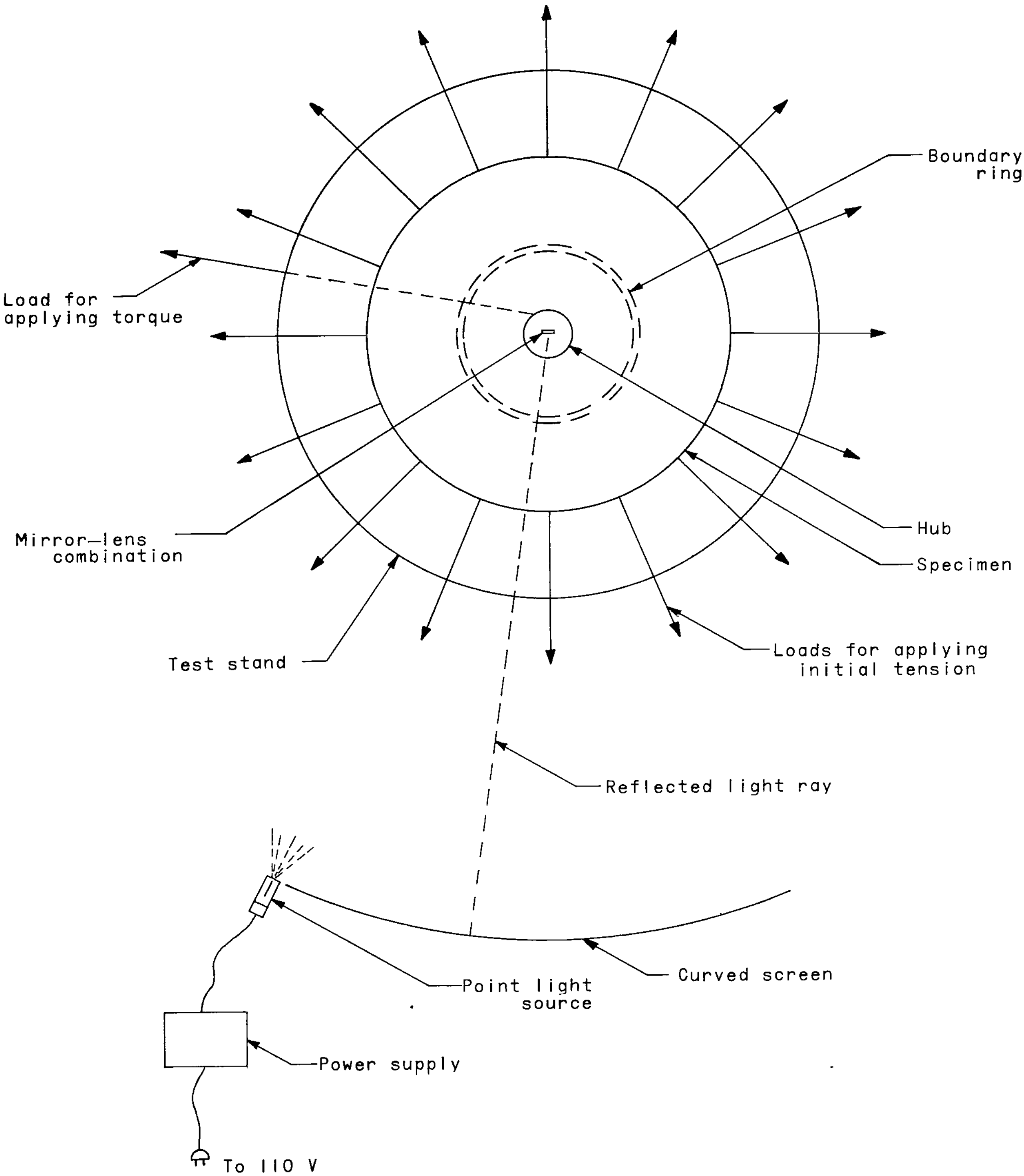

Figure 5.- Schematic of test setup.

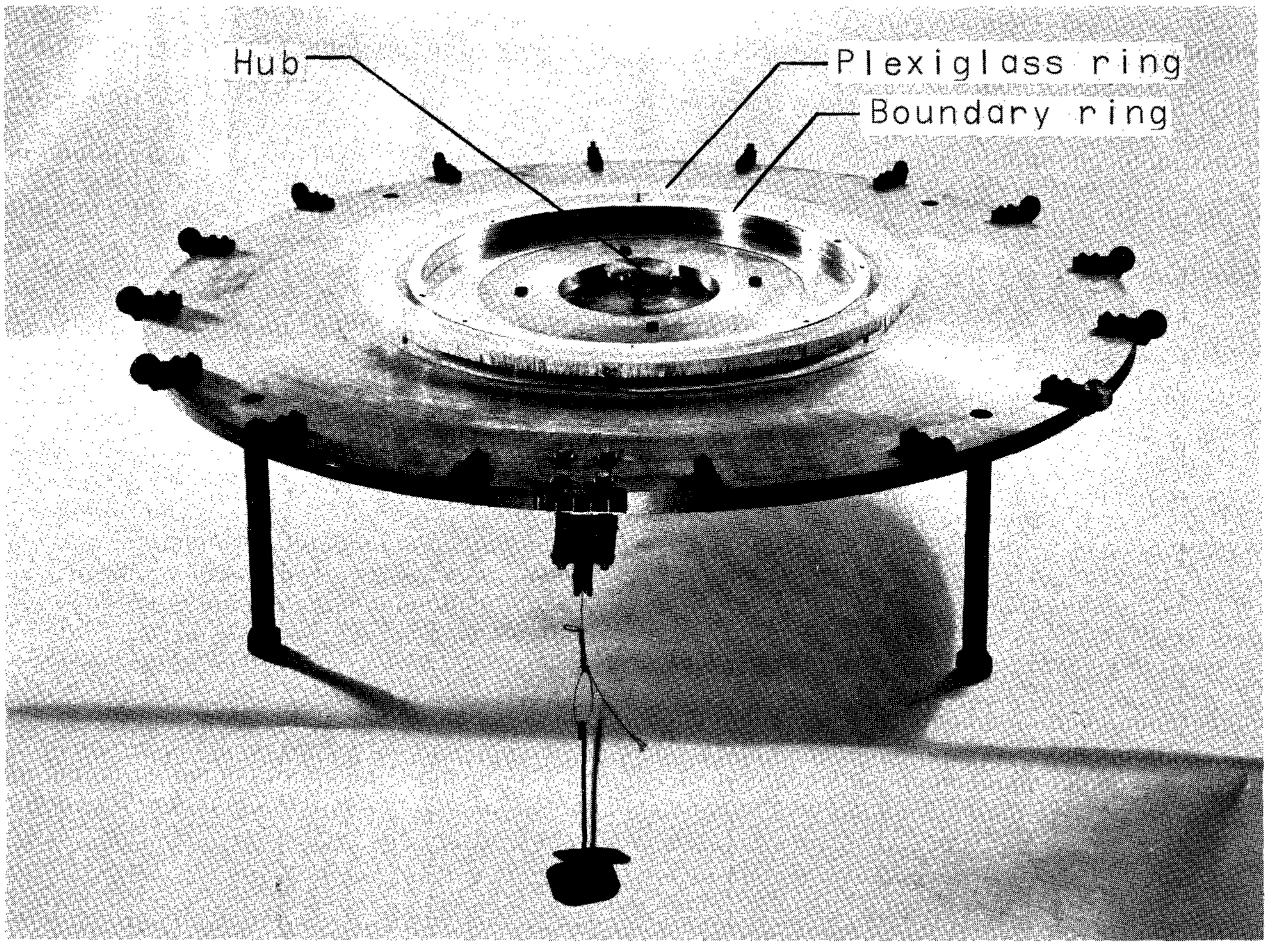

Figure 6.- Test stand without specimen. L-63-1976.1

help minimize friction in the system. A string from the pulley attached to the shaft passed over the lower knife-edge pulley (see fig. 6) to a shot bucket. The torque applied to the membrane through the hub was simply the weight of the shot bucket times the radius of the pulley attached to the shaft.

To measure the rotation of the hub, an optical lever system was utilized. A 1/4-inch (0.64-cm) square front-faced mirror with an 8-foot (2.4-m) focal-length lens glued on its face was attached to the center of the hub. This mirror-lens combination focused the reflected image of a 0.029-inch (0.074-cm) diameter point light source from a concentrated arc lamp on an 8-foot (2.4-m) radius curved screen (see fig. 5). The rotation of the hub ϕ was given by the measured displacement of the reflected image on the screen divided by twice the distance from the mirror to the screen.

More than 30 specimens were used in the development of this testing procedure. All of the test results are not presented herein because of improvements made in subsequent tests. The results of the three final tests are for

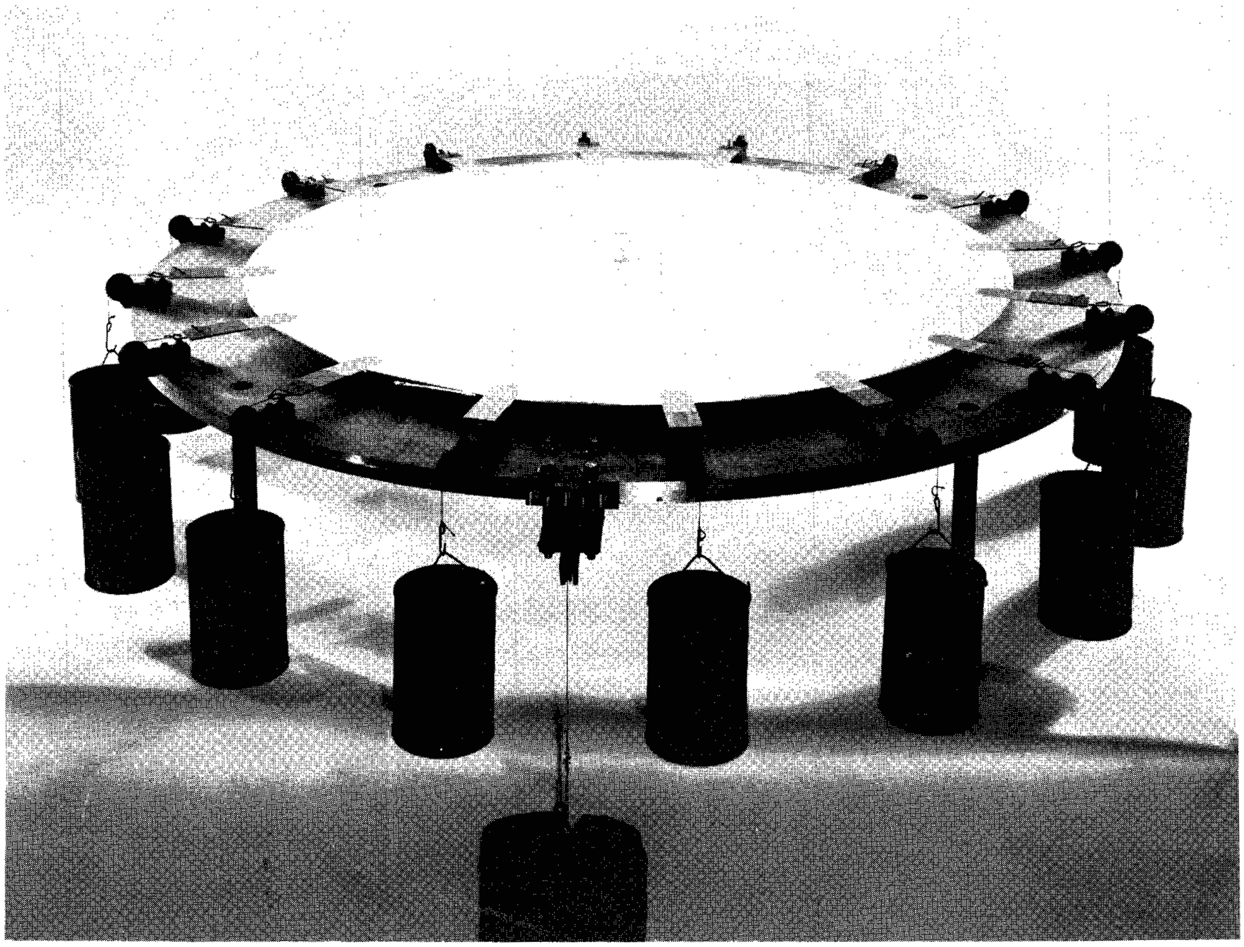

Figure 7.- Specimen mounted on test stand. L-63-1975

the same value of a/b and give essentially the same results. These results are presented as one set of test points in figure 8.

RESULTS AND DISCUSSION

In figure 3 the analytical torque-rotation plot is given for the rotation of a hub which is attached to a flat stretched membrane after the membrane is stretched. The flat stretched membrane behaves as an elastic plate until wrinkling occurs at a value of the dimensionless torque parameter $\frac{M}{2\pi a^2 Tt} = 1$. At this value of the torque parameter, a region around the hub in an elastic plate would go into compression, but since a membrane can carry no compressive stress, wrinkling occurs in the membrane. As the torque is increased, the stiffness (the slope of the torque-rotation plot) of the wrinkled stretched membrane decreases in a nonlinear fashion as shown in figure 3. However, there is

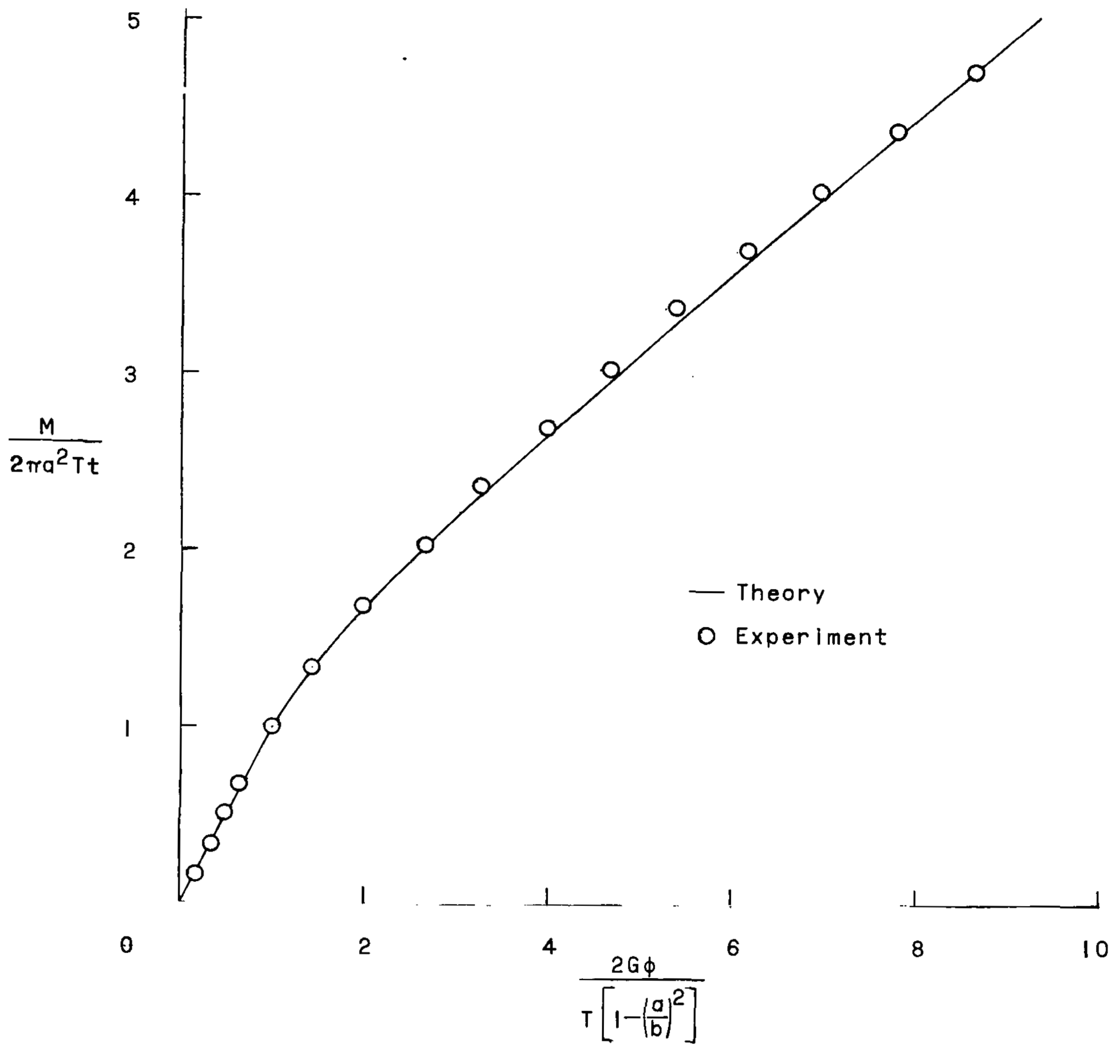

Figure 8.- Torque-rotation results for wrinkling of stretched circular membrane with hub attached after stretching; $\frac{a}{b} = \frac{5}{32}$.

no sharp reduction in stiffness and the torque can be increased until failure of the material occurs. The symbols in figure 3 denote the various values of R/a (ratio of radius of wrinkled region to radius of the hub) as indicated in the figure.

A dimensionless plot of the principal stress at the hub is given in figure 4. Here again the behavior is linear until wrinkling occurs, after which the stress increases in a nonlinear fashion.

A set of experimental points representing the results from three separate tests are given in figure 8 for a/b = 5/32. It was necessary to have the value of the shear modulus G for the material to compare the experiment with theory. The shear modulus G was obtained from the linear range of the tests, and this value for G was used throughout the wrinkled range. It is believed that this method gives a very good measure of G for the material. The value of G obtained from these experiments was 240,000 psi $\left(1.65 \frac{GN}{m^2}\right)$.

The experimental results, as can be seen in figure 8, lie slightly above the theoretical results. It is believed that some of this small discrepancy exists because the theory is for a perfect membrane with zero bending stiffness while in fact the plastic film used in the tests has a finite, though small, bending stiffness.

In appendix A the case of a hub attached to an unstretched membrane is treated. For this special case, the equations in the present paper are shown to reduce to those obtained by Reissner (ref. 2) using tension field theory.

To investigate the difference in attaching the hub to the membrane before the membrane is stretched, a discussion is given in appendix B. It was found that although the stiffness in the wrinkled range is higher for this case, wrinkling occurs earlier.

CONCLUDING REMARKS

Closed-form solutions are presented for the rotation of a hub attached to a flat stretched circular membrane for the complete range of loading from the prewrinkled state into the partly wrinkled state. Solutions are given for a membrane of finite extent for the case in which the membrane is stretched before the hub is attached, and for the case in which the membrane is unstretched. For the case of the unstretched membrane, the equations in the present paper reduce to those previously found by using tension field theory.

In the experimental study, the membrane was represented by a thin, flat plastic film. Torque-rotation studies were made on such sheets and the results from a typical test are compared with theory. The agreement between experiment and theory is very good but the experimental results lie slightly above the theory. It is believed that some of this small discrepancy exists because the plastic film in these tests was not a perfect membrane but had some finite bending stiffness.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 9, 1964.

APPENDIX A

SPECIAL CASE OF HUB ATTACHED TO UNSTRETCHED MEMBRANE

For the case of a hub attached to an unstretched membrane, wrinkling occurs over the entire membrane as soon as a finite torque is applied to the hub. The boundary conditions on displacements for this problem are $u(a) = u(b) = 0$. Applying these boundary conditions to equation (24) results in the following transcendental equation for C_4:

$$\frac{1}{\dfrac{4\pi^2t^2C_4a^2}{M^2} - 1} + \log\left(\frac{4\pi^2t^2C_4a^2}{M^2} - 1\right) - \frac{1}{\dfrac{4\pi^2t^2C_4b^2}{M^2} - 1} - \log\left(\frac{4\pi^2t^2C_4b^2}{M^2} - 1\right) = 0 \tag{A1}$$

When the constant C_4 is known, the stresses can be found from equations (17), (20), and (21).

To determine the angle of the wrinkles α (fig. 9) the stresses may be written as:

$$\left.\begin{aligned} \sigma_r &= \sigma_1 \cos^2\alpha \\ \sigma_\theta &= \sigma_1 \sin^2\alpha \\ \tau_{r\theta} &= -\sigma_1 \sin\alpha \cos\alpha \end{aligned}\right\} \tag{A2}$$

or

$$\tan^2\alpha = \frac{\sigma_\theta}{\sigma_r} = \frac{1}{\dfrac{4\pi^2t^2C_4r^2}{M^2} - 1} \tag{A3}$$

and from this equation

$$\sin\alpha = \frac{M}{2\pi t\sqrt{C_4}r} \tag{A4}$$

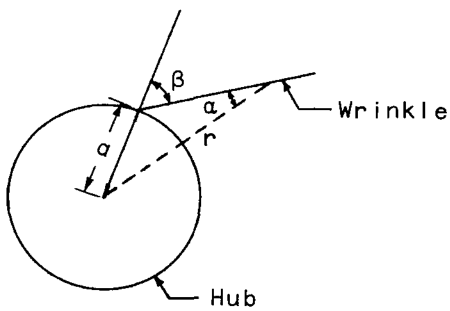

Figure 9.- Angles of wrinkle α and β.

This same problem of zero initial tension has been solved by using tension field theory by Reissner in reference 2. It will now be shown that for this case the solution in the present paper is the same as the solution found by Reissner. From figure 9 the relationship between the angles α and β is found to be:

$$\sin\alpha = \frac{a}{r}\sin\beta \tag{A5}$$

or

$$\sin\beta = \frac{M}{2\pi at\sqrt{C_4}} \tag{A6}$$

Equation (A1) can now be written as

$$\log\left(\frac{1 - \sin^2\beta}{\frac{b^2}{a^2} - \sin^2\beta}\right) + \frac{\sin^2\beta}{1 - \sin^2\beta}\left(1 - \frac{1 - \sin^2\beta}{\frac{b^2}{a^2} - \sin\beta}\right) = 0 \tag{A7}$$

From equation (A6) the angle β can be found for a given ratio of a/b. The ratio of the radial stress to shear stress may be written in terms of β as

$$\frac{\sigma_r}{\tau_{r\theta}} = -\frac{\sqrt{1 - \frac{a^2}{r^2}\sin^2\beta}}{\frac{a}{r}\sin\beta} \tag{A8}$$

At the inner and outer edges, respectively, these stress ratios are

$$\left.\begin{aligned} \frac{\sigma_r(a)}{\tau_{r\theta}(b)} &= -\cot\beta \\ \text{and} \qquad & \\ \frac{\sigma_r(b)}{\tau_{r\theta}(b)} &= -\frac{\sqrt{\frac{b^2}{a^2} - \sin^2\beta}}{\sin\beta} \end{aligned}\right\} \tag{A9}$$

Equations (A7) and (A9) are identical to those found by Reissner, and plots of β and the stress ratios as a function of values of a/b ranging from 0 to 1 are given in reference 2.

APPENDIX B

DETERMINATION OF THE EFFECT OF ATTACHING THE HUB TO THE MEMBRANE BEFORE THE MEMBRANE IS STRETCHED

The torque-rotation relationship for the case in which the hub is attached after stretching the membrane was obtained in the body of report by trial-and-error solution of equations (34), (35), and (36) and utilization of equation (37). For the case in which the hub is attached before the membrane is stretched, the boundary condition at the hub is $u(a) = 0$. The application of this boundary condition to equation (24) results in the replacement of the right-hand side of equation (34) by zero. The two equations (35) and (36) remain unchanged. These three equations are now rewritten as

$$\bar{M}\sqrt{\frac{\bar{C}_4}{\bar{M}^2} - 1}\left[\frac{1}{\frac{\bar{C}_4}{\bar{M}^2} - 1} + \frac{1}{\bar{R}^2\frac{\bar{C}_4}{\bar{M}^2} - 1} - \log\left(\frac{\bar{R}^2\frac{\bar{C}_4}{\bar{M}^2} - 1}{\frac{\bar{C}_4}{\bar{M}^2} - 1}\right) - \frac{2}{3}\right] = 0 \qquad \text{(B1)}$$

$$\bar{C}_1^{\,2} - \left[1 + 2\bar{C}_1\left(\frac{a}{b}\right)^2\right]^2\bar{R}^4 + \bar{M}^2 = 0 \qquad \text{(B2)}$$

and

$$\bar{C}_4 = \left\{\bar{R} + \left[\frac{1}{\bar{R}} + 2\bar{R}\left(\frac{a}{b}\right)^2\right]\bar{C}_1\right\}^2 + \frac{\bar{M}^2}{\bar{R}^2} \qquad \text{(B3)}$$

The quantities $\bar{M}$, $\bar{C}_1$, and $\bar{C}_4$ must be found from equations (B1), (B2), and and (B3) for given values of $\bar{R}$ and a/b by trial and error as was done previously. The hub-rotation parameter ϕ is again found from equation (37).

Note that the factor in equation (B1) which involves the radical yields no real torque-rotation relationship; therefore, the bracketed transcendental factor must be used. The solution to these equations was carried out for $a/b = 0$ (an infinite membrane) and the results are compared in figure 10 with the results for the case in which the hub was attached after stretching the membrane. As seen there is very little difference between the two solutions. Although wrinkling occurs earlier $\left(\text{at } \frac{M}{2\pi a^2 Tt} = \frac{\sqrt{3}}{2}\right)$ for the case in which the hub is attached before the membrane is stretched, a slightly stiffer torque-rotation relationship is exhibited in this case after wrinkling commences.

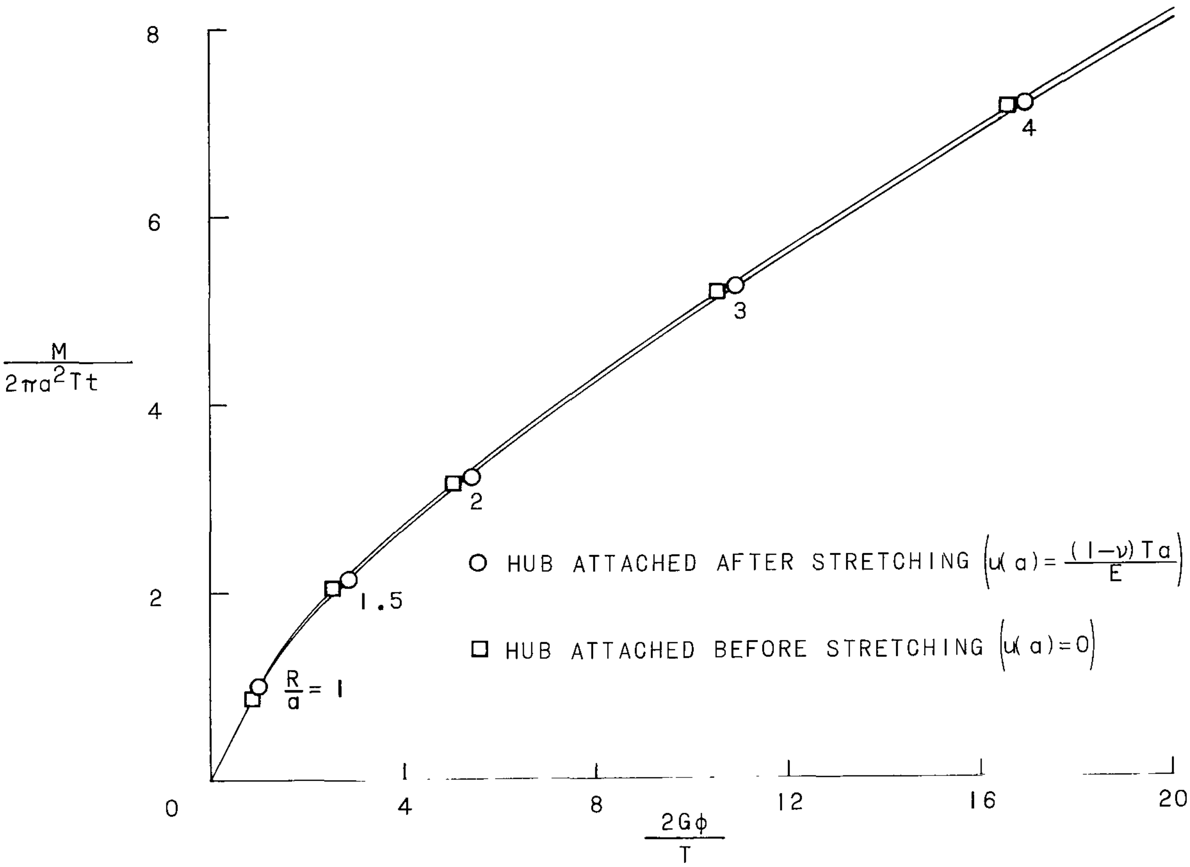

Figure 10.- Comparison of analytical results for rotation of hub attached to stretched infinite membrane for two different boundary conditions at hub.

APPENDIX C

CONVERSION OF U.S. CUSTOMARY UNITS TO SI UNITS

The International System of Units (SI) was adopted by the Eleventh General Conference on Weights and Measures, Paris, October 1960, in Resolution No. 12 (ref. 4). Conversion factors required for units used herein are:

$$\text{Length:}\quad \begin{cases} \text{Inches} \times 0.0254 = \text{Meters (m)} \\ \text{Feet} \times 0.3048 = \text{Meters} \end{cases}$$

$$\text{Force:}\quad \text{Pounds} \times 4.4482216 = \text{Newtons (N)}$$

$$\text{Stress:}\quad \text{Pounds/inch}^2 \times 6.895 \times 10^3 = \text{Newtons/meter}^2\ (\text{N/m}^2)$$

Prefixes to indicate multiples of units are:

10^{-2} centi (c)

10^{-6} micro (μ)

10^{9} giga (G)

APPENDIX D

STRESS DISTRIBUTION DUE TO AN EVEN NUMBER OF EQUALLY SPACED

RADIAL LOADS ON A CIRCULAR PLATE

An analysis is made to determine the extent of essentially uniform stress which can be obtained in a plate loaded by $N/2$ pairs of discrete equal radial in-plane loads. In reference 5 the linear elasticity solution for two diametrically opposite radial loads P applied to a circular plate is discussed. The quantity P denotes a concentrated load per unit thickness. The solution for $N/2$ pairs of diametrically opposite loads applied to a circular plate of radius r_o is obtained in the present paper by superposition of the results given in reference 5 for a single pair of loads.

For two diametrically opposite loads on a circular plate, the stresses at a point are obtained by superposing a uniform compressive stress $\sigma' = -\frac{P}{\pi r_o}$ and the two simple radial distributions at the point

$$\left.\begin{aligned} \sigma_{\rho_1} &= \frac{2P \cos \phi_1}{\pi \rho_1} \\ \sigma_{\rho_2} &= \frac{2P \cos \phi_2}{\pi \rho_2} \end{aligned}\right\} \qquad \text{(D1)}$$

The stresses σ_{ρ_1} and σ_{ρ_2} are in the directions of ρ_1 and ρ_2, respectively (see fig. 11). The stresses for one pair of radial loads $\left(\tilde{\sigma}_r,\ \tilde{\sigma}_\theta,\text{ and }\tilde{\tau}_{r\theta}\right)$ are given throughout the plate in terms of the central polar coordinates (r,θ) by the following transformation:

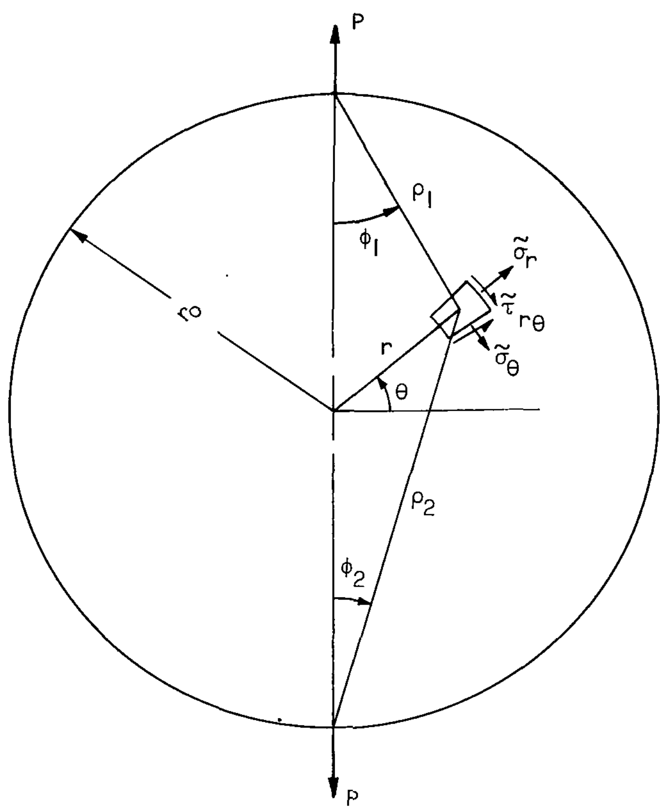

Figure 11.- Coordinate systems and stresses on an element for circular plate with a pair of diametrically opposite radial loads P.

$$\left.\begin{aligned}
\tilde{\sigma}_r &= \sigma_{\rho_1}\sin^2\psi_1 + \sigma_{\rho_2}\sin^2\psi_2 + \sigma' \\
\tilde{\sigma}_\theta &= \sigma_{\rho_1}\cos^2\psi_1 + \sigma_{\rho_2}\cos^2\psi_2 + \sigma' \\
\tilde{\tau}_{r\theta} &= -\sigma_{\rho_1}\sin\psi_1\cos\psi_1 - \sigma_{\rho_2}\sin\psi_2\cos\psi_2
\end{aligned}\right\} \tag{D2}$$

where $\psi_1 = \frac{\pi}{2} - \phi_1 + \theta$ and $\psi_2 = \frac{\pi}{2} - \phi_2 - \theta$. On combining equations (D1) and (D2) the stresses are:

$$\left.\begin{aligned}
\tilde{\sigma}_r &= \frac{2P}{\pi}\left(\frac{\cos\phi_1\sin^2\psi_1}{\rho_1} + \frac{\cos\phi_2\sin^2\psi_2}{\rho_2} - \frac{1}{2r_o}\right) \\
\tilde{\sigma}_\theta &= \frac{2P}{\pi}\left(\frac{\cos\phi_1\cos^2\psi_1}{\rho_1} + \frac{\cos\phi_2\cos^2\psi_2}{\rho_2} - \frac{1}{2r_o}\right) \\
\tilde{\tau}_{r\theta} &= -\frac{2P}{\pi}\left(\frac{\cos\phi_1\sin\psi_1\cos\psi_1}{\rho_1} + \frac{\cos\phi_2\sin\psi_2\cos\psi_2}{\rho_2}\right)
\end{aligned}\right\} \tag{D3}$$

The geometric relations between coordinate systems (ρ_1,ϕ_1), (ρ_2,ϕ_2), and (r,θ) are

$$\left.\begin{aligned}
\rho_1^{\,2} &= r_o^{\,2} + r^2 - 2rr_o\sin\theta \\
\rho_2^{\,2} &= r_o^{\,2} + r^2 + 2rr_o\sin\theta \\
\phi_1 &= \tan^{-1}\left(\frac{\frac{r}{r_o}\cos\theta}{1 - \frac{r}{r_o}\sin\theta}\right) \\
\phi_2 &= \tan^{-1}\left(\frac{\frac{r}{r_o}\cos\theta}{1 + \frac{r}{r_o}\sin\theta}\right)
\end{aligned}\right\} \tag{D4}$$

The stresses σ_r, σ_θ, and $\tau_{r\theta}$ at a point due to N/2 pairs of loads are now found by superposing the stresses for each pair of loads as given by equations (D3). These stresses may be written as:

$$\sigma_r = \sum_{k=0}^{\frac{N}{2}-1} \frac{2P}{\pi}\left[\frac{\cos \phi_{1,k} \cos^2(\theta_k - \phi_{1,k})}{\sqrt{r_o^2 + r^2 - 2rr_o \sin \theta_k}} + \frac{\cos \phi_{2,k} \cos^2(\theta_k + \phi_{2,k})}{\sqrt{r_o^2 + r^2 + 2rr_o \sin \theta_k}} - \frac{1}{2r_o}\right] \tag{D5}$$

$$\sigma_\theta = \sum_{k=0}^{\frac{N}{2}-1} \frac{2P}{\pi}\left[\frac{\cos \phi_{1,k} \sin^2(\theta_k - \phi_{1,k})}{\sqrt{r_o^2 + r^2 - 2rr_o \sin \theta_k}} + \frac{\cos \phi_{2,k} \sin^2(\theta_k + \phi_{2,k})}{\sqrt{r_o^2 + r^2 + 2rr_o \sin \theta_k}} - \frac{1}{2r_o}\right] \tag{D6}$$

$$\tau_{r\theta} = -\sum_{k=0}^{\frac{N}{2}-1} \frac{2P}{\pi}\left[\frac{\cos \phi_{1,k} \cos(\phi_{1,k} - \theta_k)\sin(\phi_{1,k} - \theta_k)}{\sqrt{r_o^2 + r^2 - 2rr_o \sin \theta_k}} + \frac{\cos \phi_{2,k} \cos(\phi_{2,k} + \theta_k)\sin(\phi_{2,k} + \theta_k)}{\sqrt{r_o^2 + r^2 + 2rr_o \sin \theta_k}}\right] \tag{D7}$$

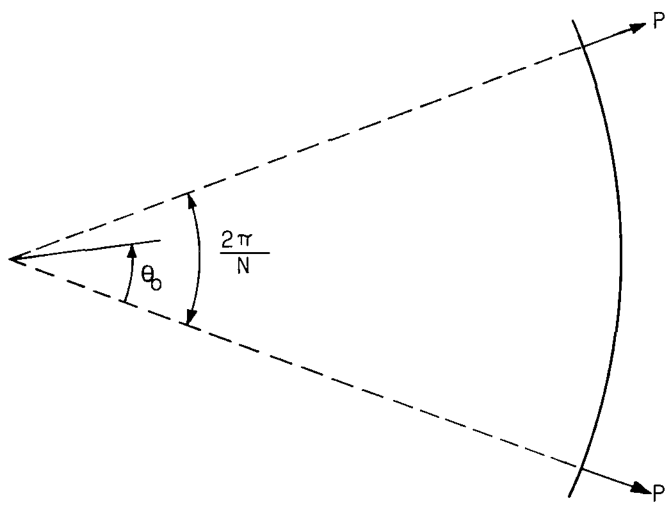

Figure 12.- Angle θ_0 between each neighboring pair of radial loads.

where $\theta_k = \theta_0 + \frac{2\pi k}{N}$, k is a a range index, and ϕ_1 and ϕ_2 are given in equations (D4). The quantity θ_0 is the angle of a diametrical line along which the stresses are calculated (see fig. 12). Since the stresses are the same between any two consecutive pairs of loads, θ_0 is defined in the region $0 \leqq \theta_0 < \frac{2\pi}{N}$. For $\theta_0 = 0$ the stresses due to N loads are obtained along a diametrical line between two loads.

Equations (D5), (D6), and (D7) were programed on a high-speed digital computer and for the case of $N = 16$ the results for three values of θ_0 are shown in figure 13. The quantity σ_{av} in figure 13 is the stress obtained if the load NP were uniformly distributed around the circumference of the circular plate. As can be seen in figure 13, the stresses for 16 discrete radial loads out to a value of $r/r_o = 0.6$ are essentially that of a circular plate subjected to a uniform stress $\sigma_{av} = 8P/\pi r_o$.

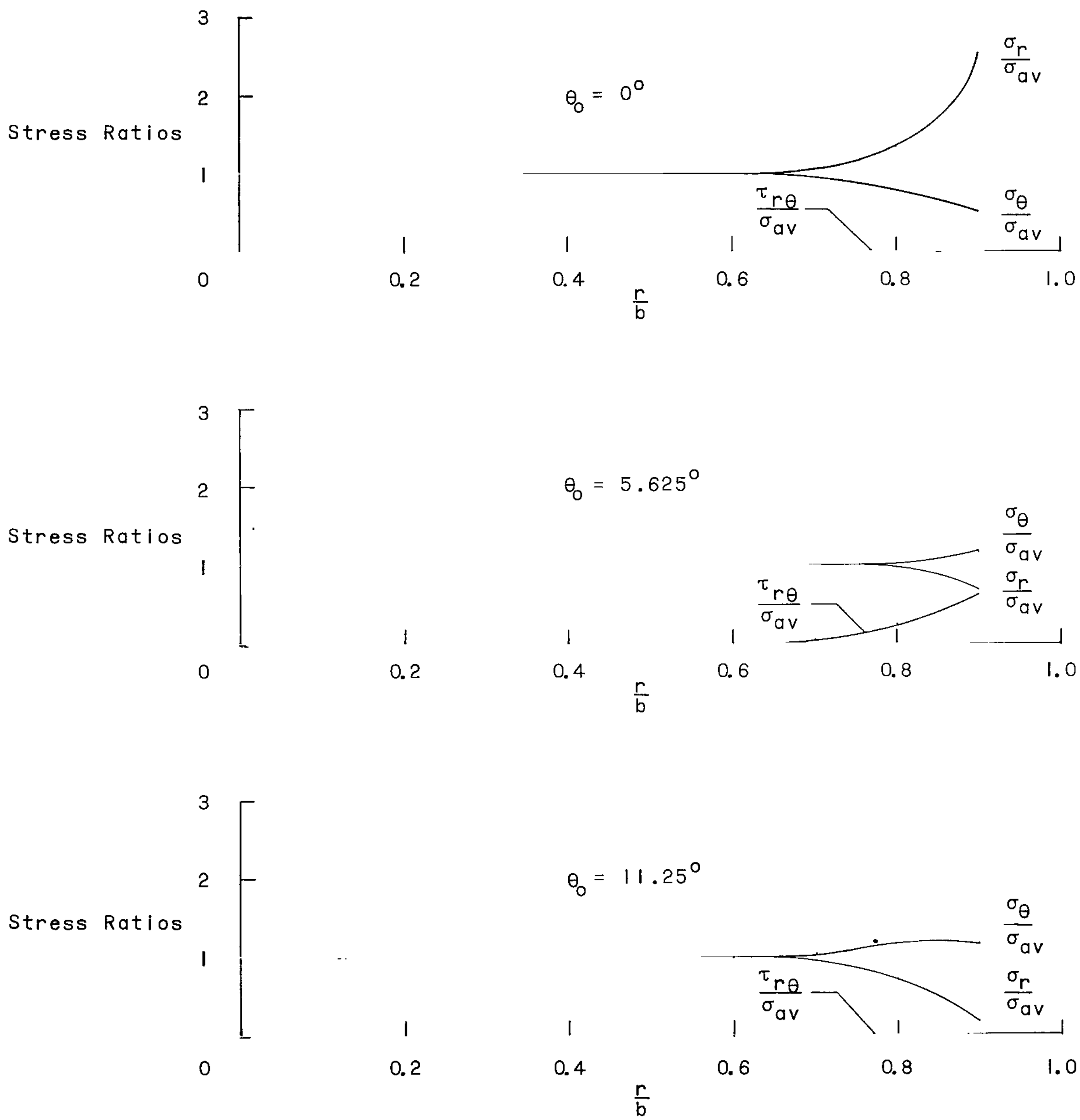

Figure 13.- Ratios of stresses in plate loaded by 16 discrete radial loads to average stress obtained by distributing discrete loads uniformly around circumference.

REFERENCES

1. Dean, W. R.: The Elastic Stability of an Annular Plate. Proc. Royal Soc., vol. 106, 1924, pp. 268-284.

2. Reissner, Eric: On Tension Field Theory. Proc. Fifth Int. Cong. Appl. Mech., John Wiley & Sons, Inc., 1938, pp. 88-92.

3. Stein, Manuel, and Hedgepeth, John M.: Analysis of Partly Wrinkled Membranes. NASA TN D-813, 1961.

4. Anon.: International System of Units, Resolution No. 12. NASA TT F-200, 1964.

5. Timoshenko, S., and Goodier, J. N.: Theory of Elasticity. Second ed., McGraw-Hill Book Co., Inc., 1951.

NASA-Langley, 1964 L-3515

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